Theorem 1. *Let $r \in \mathbb{N}$ with $r > 1$. Every natural number n can be presented uniquely in the form*

$$n = \sum_{i=0}^{m} a_i r^i$$

with $a_i < r$ for all i.

Proof. The proof will be done by induction on n. For the case $n = 0$ the only presentation obviously is

$$0 = 0 \cdot r^0$$

Now, assume the statement is true for all natural numbers below of n. First of all, we ask, what is the greatest natural number m such that

$$r^m \leq n$$

By taking the logarithm we find,

$$m = \left\lfloor \frac{\ln n}{\ln r} \right\rfloor$$

By the Euclid's division lemma there are unique $a_m, b_m \in \mathbb{N}$ with $b_m < r^m$ such that

$$n = a_m r^m + b_m$$

The definition of m trivially implies $a_m < r$. Moreover, since $b_m < r^m$ and thus $b_m < n$ we can apply the induction hypotheses to get a unique presentation

$$b_m = \sum_{i=0}^{p} a_i r_i$$

Noting that obviously $p < m$, by combining the last two equations and taking their uniqueness into account we obtain the desired presentation of n. $\square$